Algebraic Excursions in Groups and Spaces

Sommersemester 2024, Dr. Julia Heller and Kevin Klinge

Talk 1 – The p-adic numbers

Introduce the p-adic valuation and absolute value on $\mathbb{Q}$. Define the p-adic numbers $\mathbb{Q}_p$ as the completion of $\mathbb{Q}$ with respect to the p-adic absolute. Give a description of elements of $\mathbb{Q}_p$ as sequences and demonstrate how computations work. Define the topology induced on $\mathbb{Q}_p$ by the p-adic norm and give some intuition. As part of the intuition, show that $\mathbb{Q}_p$ is totally disconnected [Gou20, Proposition 2.3.9]. During the entire talk, point out the similarities and differences between $\mathbb{Q}_p$ and $\mathbb{R}$.

Remarks: It may be beneficial to at least skim the entire first four chapters from the reference, but you should focus on the topics outlined above. It will be important to boil this material down to the essentials. The main goal is to gain an intuition for $\mathbb{Q}_p$ and its topology.
References: [Gou20, Section 1.1, Section 2.3, Chapter 3, Section 4.4, Exercise 8.3]

Talk 2 – The tree for $\mathrm{SL}_2(\mathbb{Q}_p)$

Explain the construction of the tree for $\mathrm{SL}_2(k)$ for a field k with discrete valuation and show that it is a tree. Do this concretely for $k = \mathbb{Q}_p$. Elaborate on straight paths in the tree and show that every pair of edges is contained in some such path.

Remarks: When Serre says "field", he does not require that the multiplication is commutative. But you can restrict to the commutative case.
References: [Ser80, Chapter II, Sections 1.1 – 1.3], [AB08, Page 358]

Talk 3 – Lie groups

The goal of this talk is to introduce the notion of Lie groups and demonstrate explicit examples.
For that, introduce matrix Lie groups and present examples thereof, including the general and special linear groups [Hal15, Chapter 1.2.1] and [Hal15, Chapter 1.2.7]. Show that the identity component of a Lie group is connected. Define Lie groups and demonstrate that not every Lie group is a matrix Lie group [Hal15, Example 1.21]. Define one-parameter subgroups [Hal15, Definition 2.13, Theorem 2.14] and introduce needed notation, in particular that of the exponential map. Give examples. Conclude by presenting the polar decomposition of the special and general linear groups [Hal15, Proposition 2.19].

References: [Hal15, Chapters 1.1, 1.2, 1.3.1, 1.3.2, 1.5, 2.1, 2.4, 2.5]

Talk 4 – Lie algebras

The goal of this talk is to introduce Lie algebras and establish a link between Lie groups and Lie algebras. The running example will be the special linear group.
Define Lie algebras and give examples [Hal15, Chapter 3.1]. Introduce the notion of a

Lie algebra of a Lie group and determine the Lie algebras of the special and general linear groups [Hal15, Proposition 3.23]. Introduce the exponential map and explain how the Lie algebra arises as tangent space and state [Hal15, Corollary 3.47] without proof. Throughout, give examples for the special and general linear groups. See also [Hal15, Chapter 6]

Remarks: Consider the content of Talk 3.
References: [Hal15, Chapters 3.1, 3.3, 3.4, 3.7, 3.8, 3.9: Ex. 22]

Talk 5 – Algebraic groups

The goal of this talk is to introduce algebraic groups, seen as a functor from k-algebras to groups, following Appendix C of [AB08].
Focus on a detailed explanation of the material in [AB08, Chapters C.1–C.5, C.12], in particular, explain the standard examples and non-examples thereof. If suitable, survey on further properties [AB08, Chapters C.6–C.9, C.12].

References: [AB08, Appendix C] and references therein, [Gan17], [Gar10]

Talk 6 – Coxeter groups and root systems

The goal of this talk is to introduce Coxeter groups as a generalization of finite reflection groups, and root systems.
For this, define Coxeter groups and Coxeter systems following [Hum90]. Give examples of Coxeter groups, focusing on finite ones, which are reflection groups. Elaborate in particular on the symmetric group. Explain the geometric representation and the notion of roots and root systems [Hum90, Chapters 5.3, 5.4, 5.7]. See also [AB08, Appendix B], [Hal15, Theorem 7.30] and [Hal15, Chapter 8], the latter especially for the great pictures. Mention that root systems are classified [Hum90, Chapter 2] and introduce the notion of Coxeter diagram. See also [Hel19, Chapter 2.1, 2.2] for an overview.

References: [Hum90, 2, 5.1, 5.3, 5.4, 5.7], [AB08, Chapters 0.2, 2.2], [Hal15], [Hel19, Chapter 2.1, 2.2], [Bro02]

Talk 7 – Coxeter complexes and buildings

The goal of this talk is to define Coxeter complexes and buildings as simplicial complexes.
For this, define Coxeter complexes and show that they are colorable chamber complexes [AB08, Chapter 3.1]. Discuss the Coxeter complexes for the symmetric group and the one of type $\tilde{A}_2$. Define buildings [AB08, Chapter 4.1] and show that they are colorable as well. Show that infinite trees are buildings. See also [Hel19, Chapter 2.3, 3] for an overview.

References: [AB08, Chapter 3.1], [Hel19, Chapter 2.3, 3], [Bro02]

Talk 8 – BN pairs and buildings

In this talk, we will see how the geometry of a building relates to groups and vice versa [AB08, Chapter 6]. Begin with discussing the building associated to a vector space. Explain in this example how stabilizers of apartments and chambers look like. Explain the Bruhat decomposition and the notion of BN pairs. Explain the correspondence of BN pairs and buildings [AB08, Theorem 6.56]. Focus on the general linear groups. You might want to shortly survey on how this leads to the classification of linear algebraic groups.

Remarks: Depending on your pre-knowledge and personal preferences, we can consider changing the focus of the talk. Please get in touch.
References: [AB08, Chapters 4.3, 6.1, 6.2, 6.4, 6.5, 6.9] and references therein, [Bro02]

Talk 9 – Projective planes

Define incidence geometries and projective planes. Show how the Fano plane arises from the vector space $\mathbb{F}_2^3$. Show how incidences translate between spherical buildings and finite projective planes. Define the order of a projective plane. Say something about the classification of projective planes. In particular, mention that not every projective plane arises from a group. Present the argument that there are non-Desarguesian planes of prime-power order.

Remarks: You will need some background on buildings that is established in the preceding talks and can be found in [AB08, Chapter 4].
References: [AB08, page 179], [AAK$^+$18]

Talk 10 – Finite simple groups

Explain the classification theorem for finite simple groups. Explain the origins and construction of Mathieu groups. Show which Mathieu groups give rise to projective planes.

References: [Wil09, Chapter 1 and Sections 5.1 - 5.3], [Bor02]

Talk 11 – Bass-Serre theory

Define amalgamated free products. Define graphs of groups and their fundamental groups. You may assume that the graph has no loops or only briefly mention loops and HNN extensions. Show how to construct the Bass-Serre tree from a graph of groups and give the action of the fundamental group on the tree. As an example, show that $\mathrm{SL}_2(\mathbb{Z})$ is an amalgamated product of finite groups and construct the associated Bass-Serre tree.

Remarks: In [Bog08], a tree on which the fundamental group acts is constructed. This is the Bass-Serre tree even though that name is not mentioned in the reference. The relevant parts from [Ser80] are the subject of Talk 2.
References: [Bog08, Sections 2.11, 2.12, 2.16, 2.18], [BH99, Chapter II.12, Chapter C], [Bro82, Appendix to Chapter II], [Ser80]

Talk 12 – Group homology

Define group homology. Give a geometric interpretation via classifying spaces. Compute the homology of finite cyclic groups. Use this and Mayer-Vietoris for amalgamated products to compute the homology of $\mathrm{SL}_2(\mathbb{Z})$.

Remarks: Keep in mind that we will not have defined group rings or any kind of homology prior to this talk. A challenge will be to present the material in a way that is accessible to the audience.
References: [Bro82, Sections II.3, II.4, II.7]

Remarks

- Talks 3–8 are supervised by Julia, Talks 1, 2, and 9–12 are supervised by Kevin.
- Use examples to illustrate statements and proofs whenever possible. Recall that your audience is new to the topic – examples and revision of definitions help your classmates to follow and understand.
- Take some freedom in the way you present your material. Read the chapters adjacent to your main source to gain some additional background. If you wish to include something extra, or omit something, or shift the focus of your talk, please consult us first.
- Try to stick to the time frame of 60 minutes. Be aware that there may be questions from the audience during the talk. As an audience member, you are encouraged to ask questions.
- There will be some additional time for further questions, discussion, and feedback after each talk.

References

[AAK+18] Jonathan Abrunho, David Kurniadi Angdinata, Hyun Mog Kim, Yue Pan, and Xuewei Xing, *An Introduction to Finite Projective Planes*, Imperial College London, 2018. Available at `https://multramate.github.io/projects/fpp/main.pdf`.

[AB08] Peter Abramenko and Kenneth S. Brown, *Buildings*, 1st ed., Graduate Texts in Mathematics, Springer New York, NY, 2008. Available at `https://link.springer.com/book/10.1007/978-0-387-78835-7`.

[BH99] Martin R. Bridson and André Haefliger, *Metric Spacec of Non-Positive Curvature*, 1st ed., Grundlehren der mathematischen Wissenschaften, Springer Berlin, Heidelberg, 1999. Available at `https://link.springer.com/book/10.1007/978-3-662-12494-9`.

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[Gou20] Fernando Q. Gouvêa, *p-adic numbers*, 3rd ed., Universitext, Springer Cham, 2020. Available at `https://link.springer.com/book/10.1007/978-3-030-47295-5`.

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Further Reading

[Sil13] Joseph H. Silverman, *What is... the p-adic Mandelbrot Set?*, Notices Amer. Math. Soc. (2013). Available at https://www.ams.org/journals/notices/201308/rnoti-p1048.pdf?adat=September%202013&trk=2013081048&cat=whatis&galt=whatis.

[Gan17] Tom Gannon, *What is... a Scheme?*, Notices Amer. Math. Soc. (2017). Available at https://www.ams.org/journals/notices/201711/rnoti-p1300.pdf?adat=December%202017&trk=1603&cat=whatis&galt=whatis.

[Gar10] Skip Garibaldi, *What is... a Linear Algebraic Group?*, Notices Amer. Math. Soc. (2010). Available at https://www.ams.org/journals/notices/201009/rtx100901125p.pdf?adat=October%202010&trk=201009rtx100901125p&cat=whatis&galt=whatis.

[Bro02] Kenneth S. Brown, *What is... a Building?*, Notices Amer. Math. Soc. (2002). Available at https://www.ams.org/journals/notices/200210/what-is.pdf?adat=November%202002&trk=200210what-is&cat=whatis&galt=whatis.

[Bor02] Richard E. Borcherds, *What is... The Monster?*, Notices Amer. Math. Soc. (2002). Available at https://www.ams.org/journals/notices/200209/what-is.pdf?adat=October%202002&trk=200209what-is&cat=whatis&galt=whatis.

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